Homework 3 Solutions

Due: Thursday September 20th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Let $\mathcal{M}_{[0,1]}$ be the set of Lebesgue measurable subsets of $[0,1]$ and let

$$P : \mathcal{M}_{[0,1]} \longrightarrow \mathbb{R}, \quad P(A) := m^*(A)$$

be the outer measure. Consider the probability space

$$([0,1], \mathcal{M}_{[0,1]}, P)$$

(I.e the uniform probability measure on $[0,1]$). Construct $A_1, A_2, A_3, A_4 \in \mathcal{M}_{[0,1]}$ so that A_i, A_j, A_k are independent events for any distinct i, j, k but

$$P(\cap_{n=1}^4 A_n) \neq \prod_{n=1}^{4} P(A_n).$$

Solution: Let A_n be the set of numbers $x \in [0,1]$ whose nth digit after the decimal point in its binary expansion is 1 for $n = 1, 2, 3$. Define A_4 to be the set of numbers $x \in [0,1]$ with the property that the sum of the first three digits of x after the decimal point in its binary expansion is odd. Then $P(A_i) = 1/2$ for all i, $P(A_i \cap A_j) = 1/4$ for all $i \neq j$ and $P(A_i \cap A_j \cap A_k) = 1/8$ for all $i \neq j \neq k$. However

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1 \cap A_2 \cap A_3) = 1/8$$

but

$$P(A_1)P(A_2)P(A_3)P(A_4) = 1/16.$$

Problem 2: Let $f_1, \cdots, f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be measurable functions. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous. Show that

$$h : \mathbb{R} \longrightarrow \mathbb{R},\ h(x) := F(f_1(x), \cdots, f_n(x))$$

is measurable.

Solution: We have

$$F^{-1}((a,\infty)) = \bigcup_{\substack{a_1,\cdots,a_n,b_1,\cdots,b_n \in \mathbb{Q} \\ \prod_{i=1}^n (a_i,b_i) \subset F^{-1}((a,\infty))}} \prod_{i=1}^{n} (a_i, b_i).$$

Therefore

$$h^{-1}((a,\infty)) = \{x \in \mathbb{R} \ : \ (f_1(x), \cdots, f_n(x)) \in F^{-1}((a,\infty))\} =$$

$$\left\{ x \in \mathbb{R} \;:\; (f_1(x), \cdots, f_n(x)) \in \bigcup_{\substack{a_1,\cdots,a_n,b_1,\cdots,b_n \in \mathbb{Q} \\ \prod_{i=1}^n (a_i,b_i) \subset F^{-1}((a,\infty))}} \prod_{i=1}^n (a_i, b_i) \right\} =$$

$$\bigcup_{\substack{a_1,\cdots,a_n,b_1,\cdots,b_n \in \mathbb{Q} \\ \prod_{i=1}^n (a_i,b_i) \subset F^{-1}((a,\infty))}} \left(\bigcap_{i=1}^n f_i^{-1}(a_i, b_i) \right).$$

Since this is a countable union of intersections of measurable sets we get that $h^{-1}((a,\infty))$ is measurable and hence h is measurable.

Problem 3: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and $E \subset \mathbb{R}$ a null set with the property that for each $x \in \mathbb{R} - E$ and each $\epsilon > 0$ there exists $\delta > 0$ so that $|f(y) - f(x)| < \epsilon$ for all $y \in \mathbb{R} - E$ satisfying $|x - y| < \delta$.

Show that f is measurable.

Solution: For each $x \in f^{-1}((a,\infty)) \cap (\mathbb{R} - E)$ choose $\delta_x > 0$ so that $|f(y) - f(x)| < |x - a|$ for each $y \in \mathbb{R} - E$ satisfying $|y - x| < \delta_x$. Define

$$O := \cup_{x \in f^{-1}((a,\infty))} (x - \delta_x, x + \delta_x).$$

Then $O \cap (\mathbb{R} - E) = f^{-1}((a,\infty)) \cap (\mathbb{R} - E)$ and hence

$$f^{-1}((a,\infty)) = (O \cap (\mathbb{R} - E)) \cup (f^{-1}((a,\infty)) \cap E). \tag{1}$$

Now O is open, and hence it is measurable. Also any subset of the null set E is null and hence measurable. Therefore by Equation (1), $f^{-1}((a,\infty))$ is measurable for each $a \in \mathbb{R}$. Hence f is measurable.

Problem 4: Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is measurable iff $f^{-1}(B)$ is measurable for each Borel set B.

Solution: If $f^{-1}(B)$ is measurable for each Borel set B then it is clear that f is measurable since intervals are Borel sets. Now suppose that f is measurable. Let $\mathcal{B} \subset 2^{\mathbb{R}}$ be the set of Borel sets and $\mathcal{M} \subset 2^{\mathbb{R}}$ the set of all measurable sets. Define

$$Q := \{B \in \mathcal{B} \;:\; f^{-1}(B) \in \mathcal{M}\} \subset \mathcal{B}.$$

Then Q is a σ-field because

(a) $\mathbb{R} = f^{-1}(\mathbb{R}) \in \mathcal{M}$ and hence $\mathbb{R} \in Q$.

(b) If $B \in Q$ then $f^{-1}(B^c) = (f^{-1}(B))^c \in \mathcal{M}$ since $f^{-1}(B) \in \mathcal{M}$. Hence $B^c \in Q$.

(c) If $(B_n)_{n\in\mathbb{N}} \in Q$ then $f^{-1}(\cup_{n\in\mathbb{N}} B_n) = \cup_{n\in\mathbb{N}} f^{-1}(B_n) \in \mathcal{M}$ since $f^{-1}(B_n) \in \mathcal{M}$ for each $n \in \mathbb{N}$. Hence $\cup_{n\in\mathbb{N}} B_n \in Q$.

Also Q contains each interval I since f is measurable. Hence Q is a σ-field containing every interval and hence $\mathcal{B} \subset Q$. Hence $Q = \mathcal{B}$ and so $f^{-1}(B)$ is measurable for each $B \in \mathcal{B}$.